The Motion of a Solid in Infinite Liquid.

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The present paper is a sequel to the one with the same title in Vol. XX, of the *American Journal of Mathematics*, 1897; it carries on the investigation for the external shape of the body shown by Clebsch in *Mathematische Annalen* III, p. 238, to lead to an elliptic function solution of the same character as before in the simple shape of revolution considered by Kirchhoff in his *Vorlesungen*.

Contrary to anticipation derived from a study of Halphen's treatment of the same problem in his *Fonctions elliptiques* II, Chap. 4, of which this article may be considered a commentary, the extension from Kirchhoff's shape to the more general form discussed by Clebsch does not introduce a complication essentially greater. Moreover the extra constant at disposal enables us to construct an algebraical case of the motion with greater ease, much as the discussion of the symmetrical top is in many respects simpler in its analysis than the more restricted case of the Spherical Pendulum.

The elliptic-function solution of the motion under no force of a solid in infinite liquid is then the object of this memoir; for the more general case expressible by the double-theta hyperelliptic function, initiated by H. Weber, the *Fortschritte der Mathematik* must be consulted for references to the discussion of various authors, such as H. Weber, F. Kötter, R. Liouville, Caspary, Jukovsky, Liaponoff, and others.

1. We employ the notation of Halphen at the outset, with the occasional introduction of a symbol from the paper in the *American Journal*; and now with Clebsch's extended form of the kinetic energy for helicoidal symmetry

$$(1)\qquad \left\{\begin{aligned} T &= \tfrac{1}{2}p(x_1^2+x_2^2)+\tfrac{1}{2}p'x_3^2\\ &\quad + q(x_1y_1+x_2y_2)+q'x_3y_3\\ &\quad + \tfrac{1}{2}r(y_1^2+y_2^2)+\tfrac{1}{2}r'y_3^2,\end{aligned}\right.$$

and the dynamical equations

$$(2)\qquad \frac{dx_1}{dt}=x_3\frac{\partial T}{\partial y_2}-x_2\frac{\partial T}{\partial y_3},\;\ldots\ldots,\;\ldots\ldots,$$

$$(3)\qquad \frac{dy_1}{dt}=-x_3\frac{\partial T}{\partial x_2}+x_2\frac{\partial T}{\partial x_3}-y_3\frac{\partial T}{\partial y_2}+y_2\frac{\partial T}{\partial y_3},\;\ldots\ldots,\;\ldots\ldots$$

we arrive immediately at the three integrals

$$(4)\qquad 2T=\text{constant}=l,$$

$$(5)\qquad x_1^2+x_2^2+x_3^2=\text{constant}=m,$$

$$(6)\qquad x_1y_1+x_2y_2+x_3y_3=\text{constant}=n,$$

suppose; and in addition

$$(7)\qquad \frac{dy_3}{dt}=0,\text{ so that } y_3 \text{ is constant.}$$

Denoting the component linear and angular velocity with respect to axes OA, OB, OC, fixed in the body by

$$(8)\qquad U,\ V,\ W,\ P,\ Q,\ R,$$

as in Halphen, but a change to capital letters from the notation in the *American Journal*,

$$(9)\qquad U=\frac{\partial T}{\partial x_1}=px_1+qy_1,$$

$$(10)\qquad V=\frac{\partial T}{\partial x_2}=px_2+qy_2,$$

$$(11)\qquad W=\frac{\partial T}{\partial x_3}=p'x_3+q'y_3.$$

Introducing Euler's unsymmetrical angles θ, ϕ, ψ defining the position of OA, OB, OC with respect to axes OX, OY, OZ having fixed direction in space, on the system employed in Klein-Sommerfeld, *Theorie des Kreisels*, p. 19, in accordance with figure 1, and the scheme

	A	B	C
X	$\cos\phi\cos\psi - \cos\vartheta\sin\phi\sin\psi$	$-\sin\phi\cos\psi - \cos\vartheta\cos\phi\sin\psi$	$\sin\vartheta\sin\psi$
Y	$\cos\phi\sin\psi + \cos\vartheta\sin\phi\cos\psi$	$-\sin\phi\sin\psi + \cos\vartheta\cos\phi\cos\psi$	$-\sin\vartheta\cos\psi$
Z	$\sin\vartheta\sin\phi$	$\sin\vartheta\cos\phi$	$\cos\vartheta$

$$\frac{\partial T}{\partial y_1} = qx_1 + ry_1 = P = \cos\phi\frac{d\vartheta}{dt} + \sin\vartheta\sin\phi\frac{d\psi}{dt}, \tag{12}$$

$$\frac{\partial T}{\partial y_2} = qx_2 + ry_2 = Q = -\sin\phi\frac{d\vartheta}{dt} + \sin\vartheta\cos\phi\frac{d\psi}{dt}, \tag{13}$$

so that

$$q(x_1 + x_2 i) + r(y_1 + y_2 i) = P + Qi = \left(\frac{d\vartheta}{dt} + i\sin\vartheta\frac{d\psi}{dt}\right)e^{-\phi i}; \tag{14}$$

$$\frac{\partial T}{\partial y_3} = q'x_3 + r'y_3 = R = \frac{d\phi}{dt} + \cos\vartheta\frac{d\psi}{dt}. \tag{15}$$

From the third equation of system (2)

$$\left\{\begin{aligned} \frac{dx_3}{dt} &= x_2\frac{\partial T}{\partial y_1} - x_1\frac{\partial T}{\partial y_2} \\ &= x_2(qx_1 + ry_1) - x_1(qx_2 + ry_2) \\ &= r(x_2 y_1 - x_1 y_2), \end{aligned}\right. \tag{16}$$

and from (6), distinguishing this n by an accent,

$$x_1 y_1 + x_2 y_2 = n' - x_3 y_3, \tag{17}$$

so that

$$\frac{1}{r^2}\left(\frac{dx_3}{dt}\right)^2 = (x_2 y_1 - x_1 y_2)^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1 y_1 + x_2 y_2)^2 \tag{18}$$

Again from (5) and (4)

(19) $$x_1^2 + x_2^2 = m - x_3^2$$

(20) $$\begin{cases} r(y_1^2 + y_2^2) = l - p(m - x_3^2) - p'x_3^2 \\ \qquad - 2q(n' - x_3y_3) - 2q'x_3y_3 - r'y_3^2 \\ \qquad = (p - p')x_3^2 + 2(q - q')x_3y_3 - mp - 2n'q - r'y_3^2 + l \\ \qquad = (p - p')(x_3^2 + hx_3 - m_1) \end{cases}$$

on putting

(21) $$(p - p')m_1 = mp + 2n'q + r'y_3^2 - l,$$

(22) $$(p - p')h = 2(q - q')y_3;$$

and now

(23) $$\left(\frac{dx_3}{dt}\right)^2 = r(p' - p)(x_3^2 - m)(x_3^2 + hx_3 - m_1) - r^2(x_3y_3 - n')^2,$$

a quartic in x_3, as in Halphen's (13), so that x_3 is an elliptic function of the time t.

Writing F^2 for m, as in the *American Journal* (*Am. J. M.*), then from Halphen's equation (45), F. E. II, p. 159,

(24) $$x_1 = F \cos AZ = F \sin \vartheta \sin \phi,$$

(25) $$x_2 = F \cos BZ = F \sin \vartheta \cos \phi;$$

and introducing Klein's $\alpha, \beta, \gamma, \delta$ defined by

(26) $$\begin{aligned} \alpha &= \cos \tfrac{1}{2}\vartheta e^{\frac{1}{2}(\phi+\psi)i}, & \beta &= i \sin \tfrac{1}{2}\vartheta e^{\frac{1}{2}(-\phi+\psi)i}, \\ \gamma &= i \sin \tfrac{1}{2}\vartheta e^{\frac{1}{2}(\phi-\psi)i}, & \delta &= \cos \tfrac{1}{2}\vartheta e^{\frac{1}{2}(-\phi-\psi)i}, \end{aligned}$$

(27) $$x_1 + x_2 i = F(\cos AZ + i \cos BZ) = iF \sin \vartheta e^{-\phi i} = 2F\beta\delta,$$

(28) $$x_3 = F \cos CZ = F \cos \vartheta = F(\alpha\delta + \beta\gamma);$$

thus expressing the constancy in magnitude and direction of the resultant momentum F of the system, taken as acting in the direction OZ.

Writing z for $\cos \vartheta$, equation (23) becomes

(29) $$\left(\frac{dz}{dt}\right)^2 = F^2 r(p' - p)(z^2 - 1)\left(z^2 + \frac{h}{F} z - \frac{m_1}{F^2}\right) - \left(\frac{Fry_3z - n'r}{F}\right)^2,$$

and putting

(30) $$F^2 r(p' \sim p) = n^2, \; F^2 r(p' - p) = an^2,$$

so that

(31) $a = +1$ when $p' - p$ is positive, as for prolate bodies,

(32) $a = -1$ . . . . negative, ... oblate . . . .

then

(33) $$\left(\frac{dz}{dt}\right)^2 = n^2 Z,$$

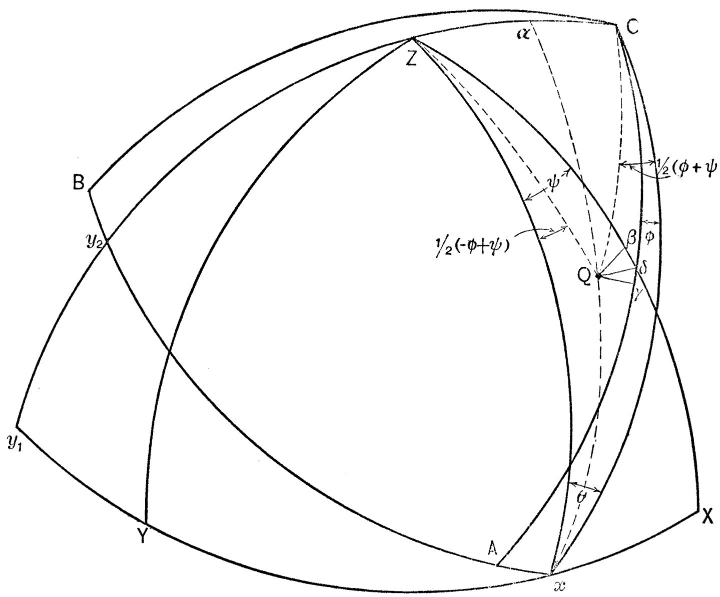
where

(34) $$Z = a(z^2-1)\left(z^2+\frac{h}{F}z-\frac{m_1}{F^2}\right)-\left(\frac{Fry_3 z - n'r}{Fn}\right)^2,$$

and

(35) $$nt+\varepsilon = \int \frac{dz}{\sqrt{Z}},$$

an elliptic integral of the I kind.



2. The system of axes (X, Y, Z) can be brought into coincidence with the system (A, B, C) either by the successive rotation through the Eulerian angles ψ, ϑ, ϕ, about the axes OZ, Ox, OC in succession, as shown in figure 1; or else by a single rotation about an axis OQ, through an angle ω suppose.

Then if a, b, c denote the angles which OQ makes with the axes (X, Y, Z) or (A, B, C), the quaternion versor Q which performs the displacement of (X, Y, Z) into (A, B, C) is given by

(1) $$Q = Ai + Bj + Ck + D,$$

where

(2) $$A = \sin\tfrac{1}{2}\omega\cos a,\ B = \sin\tfrac{1}{2}\omega\cos b,\ C = \sin\tfrac{1}{2}\omega\cos c,\ D = \cos\tfrac{1}{2}\omega.$$

Now in the figure, Q lies on ax, the bisector of the angle $ZxC = \vartheta$, so that $QZx = QCx$; and drawing the perpendiculars from Q, $Q\beta$ on ZX, $Q\gamma$ on CA, ZX and CA intersecting in δ, the spherical triangles $QZ\beta$, $QC\gamma$ are congruent, and $QZ\beta = QC\gamma$. Thus

(3) $$QZx = QCx = \text{half sum of } XZx,\ ACx = \tfrac{1}{2}(\ \ \phi + \psi),$$

(4) $$QZ\delta = QC\delta = \text{half difference, } \ldots\ldots = \tfrac{1}{2}(-\phi + \psi);$$

and thence

(5) $$D = \cos \tfrac{1}{2}\omega \qquad = \cos \alpha QZ \qquad = \cos \tfrac{1}{2}\vartheta \cos \tfrac{1}{2}(\phi + \psi),$$

(6) $$C = \sin \tfrac{1}{2}\omega \cos c = \cos \alpha Z \cos \alpha ZQ = \cos \tfrac{1}{2}\vartheta \sin \tfrac{1}{2}(\phi + \psi).$$

Similarly

(7) $$A = \sin \tfrac{1}{2}\vartheta \cos \tfrac{1}{2}(-\phi + \psi),\ B = \sin \tfrac{1}{2}\vartheta \sin \tfrac{1}{2}(-\phi + \psi);$$

and thus

(8) $$A = \frac{\beta + \gamma}{2i},\quad B = \frac{-\beta + \gamma}{2},\quad C = \frac{\alpha - \delta}{2i},\quad D = \frac{\alpha + \delta}{2},$$

(9) $$\alpha = D + Ci,\ \beta = Ai - B,\quad \gamma = Ai + B,\ \delta = D - Ci,$$

(Klein-Sommerfeld *Kreisel theorie*, p. 21).

In Routh's *Rigid Dynamics* the sequence of displacement is ψ about OZ and then ϑ about Oy_1, leading to a result of appearance slightly different.

3. The relations

(1) $$x_1 + x_2 i = \frac{r(U + Vi) - q(P + Qi)}{pr - q^2},\quad x_3 = \frac{r'W - q'R}{p'r' - q'^2},$$

(2) $$y_1 + y_2 i = \frac{-q(U + Vi) + p(P + Qi)}{pr - q^2},\quad y_3 = \frac{-q'W + p'R}{p'r' - q'^2},$$

obtainable from (9), (10), (11), (14), (15), §1, give $x_1, x_2, x_3, y_1, y_2, y_3$ in terms of the linear and angular components of velocity, and hence the kinetic energy expressed as a quadratic function of U, V, W, P, Q, R becomes

(3) $$\left\{\begin{aligned} T &= \tfrac{1}{2}\frac{r(U^2 + V^2) - 2q(UP + VQ) + p(P^2 + Q^2)}{pr' - q^2} \\ &+ \tfrac{1}{2}\frac{r'W^2 - 2q'WR + p'R^2}{p'r' - q'^2} \end{aligned}\right.$$

as in the *American Journal of Mathematics* where q and q' are zero; and now

the six dynamical equations of Kirchhoff become

(4) $$\frac{d}{dt}\frac{\partial T}{\partial U} - R\frac{\partial T}{\partial V} + Q\frac{\partial T}{\partial W} = 0, \ \ldots, \ \ldots$$

(5) $$\frac{d}{dt}\frac{\partial T}{\partial P} - R\frac{\partial T}{\partial Q} + Q\frac{\partial T}{\partial R} - W\frac{\partial T}{\partial V} + V\frac{\partial T}{\partial W} = 0, \ \ldots, \ \ldots$$

having three first integrals, and leading to the same result as before, but not so simply and directly.

4. Darboux's notation employed in the corresponding motion of the top (Despeyrous—*Mécanique*) is convenient here also; using L' instead of his B, we put

(1) $$\frac{n'r}{Fn} = 2\frac{L}{M}, \quad \frac{Fry_3}{Fn} = 2\frac{L'}{M}, \quad \frac{h}{F} = -4\frac{B}{M},$$

and now

(2) $$\begin{cases} Z = a(z^2-1)\left(z^2 - 4\frac{B}{M}z - \frac{m_1}{m}\right) - 4\left(\frac{L'z - L}{M}\right)^2 \\ \quad = a(z^2-1)\left(z^2 - 4\frac{B}{M}z - 1 - aD\right) - 4\left(\frac{L'z - L}{M}\right)^2, \end{cases}$$

or

(3) $$\begin{cases} Z = a(z^2-1)\left(z^2 - 4\frac{B}{M}z - \frac{m_2}{m}\right) - 4\left(\frac{Lz - L'}{M}\right)^2 \\ \quad = a(z^2-1)\left(z^2 - 4\frac{B}{M}z - 1 - aE\right) - 4\left(\frac{Lz - L'}{M}\right)^2, \end{cases}$$

where

(4) $$1 + aD = \frac{m_1}{m}, \quad 1 + aE = \frac{m_2}{m},$$

(5) $$a\frac{m_1 - m_2}{m} = D - E = 4\frac{L^2 - L'^2}{M^2}.$$

Putting

(6) $$Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e,$$

then by analogy with the corresponding case of the spinning top

(7) $$\begin{cases} Z = a(z^2-1)(z^2 + 4abz - 1 - aD) - 4\left(\frac{L'z - L}{M}\right)^2 \\ \quad = a(z^2-1)(z^2 + 4abz - 1 - aE) - 4\left(\frac{Lz - L'}{M}\right)^2, \end{cases}$$

where

$$(8) \qquad D + 4\frac{L'^2}{M^2} = E + 4\frac{L^2}{M^2} = F' \text{ suppose}$$

and

$$(9) \qquad \begin{cases} D = a\left(\frac{m_1}{m} - 1\right),\ E = a\left(\frac{m_2}{m} - 1\right), \\ ab = -\frac{B}{M},\ b = -a\frac{B}{M}. \end{cases}$$

The special case of h, b, $B = 0$ was the one considered by Kirchhoff, and in the *American Journal of Mathematics*.

But now we continue with the more general case invented by Clebsch and discussed by Halphen, and identify the notation as we go on.

Proceeding with the determination of the Eulerian angles, ϑ, ϕ, ψ, by means of the Table above,

$$(10) \qquad \cos AX + i\cos AY = (\cos\phi + i\cos\vartheta\sin\phi)e^{\psi i} = \alpha^2 - \beta^2,$$

$$(11) \qquad \cos AZ = \sin\vartheta\sin\phi = \beta\delta - \alpha\gamma,$$

$$(12) \qquad \cos BX + i\cos BY = (-\sin\phi + i\cos\vartheta\cos\phi)e^{\psi i} = i(\alpha^2 + \beta^2),$$

$$(13) \qquad \cos BZ = \sin\vartheta\cos\phi = i(-\beta\delta - \alpha\gamma),$$

$$(14) \qquad \cos CX + i\cos CY = -i\sin\vartheta e^{\psi i} = -2\alpha\beta,$$

$$(15) \qquad \cos CZ = \cos\vartheta = \alpha\delta + \beta\gamma.$$

So also

$$(16) \qquad \cos AX + i\cos BX = (\cos\psi - i\cos\vartheta\sin\psi)e^{-\phi i} = -\beta^2 + \delta^2,$$

$$(17) \qquad \cos CX = \sin\vartheta\sin\psi = -\alpha\beta + \gamma\delta,$$

$$(18) \qquad \cos AY + i\cos BY = (\sin\psi + i\cos\vartheta\cos\psi)e^{-\phi i} = i(\beta^2 + \delta^2),$$

$$(19) \qquad \cos CY = -\sin\vartheta\cos\psi = i(\alpha\beta + \gamma\delta),$$

$$(20) \qquad \cos AZ + i\cos BZ = i\sin\vartheta e^{-\phi i} = 2\beta\delta,$$

$$(21) \qquad \cos CZ = \cos\vartheta = \alpha\delta + \beta\gamma.$$

Again

$$(22) \qquad R = 2i\left(\alpha\frac{d\delta}{dt} - \frac{d\beta}{dt}\gamma\right),$$

$$(23) \qquad P + Qi = \left(\frac{d\vartheta}{dt} + i\sin\vartheta\frac{d\psi}{dt}\right)e^{-\phi i} = 2i\left(\beta\frac{d\delta}{dt} - \frac{d\beta}{dt}\delta\right),$$

$$(24)\qquad \begin{cases} r(y_1+y_2 i) = P + Qi - q(x_1+x_2 i) \\ \qquad = \left[\dfrac{d\vartheta}{dt} + i\sin\vartheta\left(\dfrac{d\psi}{dt} - qF\right)\right] e^{-\phi i} \\ \qquad = 2i\left(\beta\dfrac{d\delta}{dt} - \dfrac{d\beta}{dt}\delta\right) - 2qF\beta\delta, \end{cases}$$

$$(25)\qquad ry_1 = \cos\phi\frac{d\vartheta}{dt} + \sin\vartheta\sin\phi\left(\frac{d\psi}{dt} - qF\right),$$

$$(26)\qquad ry_2 = -\sin\phi\frac{d\vartheta}{dt} + \sin\vartheta\cos\phi\left(\frac{d\psi}{dt} - qF\right),$$

$$(27)\qquad \begin{cases} n' - x_3 y_3 = x_1 y_1 + x_2 y_2 \\ \qquad = F\sin\vartheta(y_1\sin\phi + y_2\cos\phi) \\ \qquad = \dfrac{F}{r}\sin^2\vartheta\left(\dfrac{d\psi}{dt} - qF\right), \end{cases}$$

so that, with $x_3 = Fz$,

$$(28)\qquad \frac{d\psi}{dt} - qF = \frac{n'r - Fry_3 z}{F(1-z^2)} = 2n\frac{L - L'z}{M(1-z^2)},$$

$$(29)\qquad \psi - qFt = 2\int\frac{L - L'z}{M(1-z^2)}\frac{dz}{\sqrt{Z}},$$

introducing the III elliptic integral.

From (15), §1,

$$(30)\qquad \begin{cases} \dfrac{d\phi}{dt} = R - \cos\vartheta\dfrac{d\psi}{dt} \\ \quad = q'x_3 + r'y_3 - z\left(qF + \dfrac{2n}{M}\dfrac{L - L'z}{1-z^2}\right) \\ \quad = (q'-q)Fz + (r'-r)y_3 + 2n\dfrac{L'}{M} - 2\dfrac{n}{M}\dfrac{Lz - L'z^2}{1-z^2} \\ \quad = (q'-q)Fz + (r'-r)y_3 + 2n\dfrac{L' - Lz}{M(1-z^2)} \end{cases}$$

$$(31)\qquad \phi = (q'-q)\frac{F}{n}\int\left(z + \frac{r'-r}{q'-q}\frac{y_3}{F}\right)\frac{dz}{\sqrt{Z}} + 2\int\frac{L' - Lz}{M(1-z^2)}\frac{dz}{\sqrt{Z}},$$

introducing another III elliptic integral, with parameter (w) corresponding to $z = \infty$.

Thence

$$(32)\qquad \phi + \psi = \frac{F}{n}\int\left[(q'-q)z + (r'-r)\frac{y_3}{F} + q\right]\frac{dz}{\sqrt{Z}} + 2\int\frac{L + L'}{M(1+z)}\frac{dz}{\sqrt{Z}},$$

$$(33)\qquad \phi - \psi = \frac{F}{n}\int\left[(q'-q)z + (r'-r)\frac{y_3}{F} - q\right]\frac{dz}{\sqrt{Z}} - 2\int\frac{L - L'}{M(1-z)}\frac{dz}{\sqrt{Z}}.$$

From (24) and (27), §1,

$$(34) \qquad \frac{y_1 + y_2 i}{x_1 + x_2 i} = \frac{\dfrac{d\vartheta}{dt} + i \sin\vartheta\left(\dfrac{d\psi}{dt} - qF\right)}{iFr\sin\vartheta}$$

and thence Halphen's Φ (F. E. II, p. 158), employing his notation for ρ, s for a moment, is given by

$$(35) \qquad \left\{\begin{aligned} \frac{\rho^2}{s} y_3 \Phi &= (x_1^2 + x_2^2)\frac{y_1 + y_2 i}{x_1 + x_2 i} \\ &= F^2 \sin^2\vartheta \frac{-i\dfrac{d\vartheta}{dt} + \sin\vartheta\left(\dfrac{d\psi}{dt} - qF\right)}{Fr\sin\vartheta} \\ &= \frac{F}{r}\left(i\frac{dz}{dt} + 2n\frac{L - L'z}{M}\right) \\ &= \frac{Fn}{r}\left(i\sqrt{Z} + 2\frac{L - L'z}{M}\right). \end{aligned}\right.$$

5. For the motion of translation, denote by X, Y, Z, the coordinates of the origin O fixed in the body, with respect to axes fixed in space and parallel to OX, OY, OZ; then according to Kirchhoff's equations (Halphen, F. E. II, p. 162),

$$(1) \qquad FX = y_1 \cos AY + y_2 \cos BY + y_3 \cos CY,$$

$$(2) \qquad FY = -y_1 \cos AX - y_2 \cos BX - y_3 \cos CX;$$

so that, from the preceding relations

$$(3) \qquad \begin{aligned} F(X + Yi) = -i\,[&y_1(\cos\phi + i\cos\vartheta\sin\phi) + \\ &y_2(-\sin\phi + i\cos\vartheta\cos\phi) - iy_3\sin\vartheta]e^{\psi i} \end{aligned}$$

$$(4) \qquad \left\{\begin{aligned} Fr(X + Yi) &= \left[-i\frac{d\vartheta}{dt} + \sin\vartheta\cos\vartheta\left(\frac{d\psi}{dt} - qF\right) - 2n\frac{L'}{M}\sin\vartheta\right]e^{\psi i} \\ &= \left(-i\frac{d\vartheta}{dt} + 2n\frac{Lz - L'}{M\sin\vartheta}\right)e^{\psi i}. \end{aligned}\right.$$

Changing to polar coordinates ρ, ϖ in a plane perpendicular to OZ, such that

$$(5) \qquad \frac{Fr}{n}X = \rho\cos\varpi, \quad \frac{Fr}{n}Y = \rho\sin\varpi,$$

$$(6) \qquad \rho\exp(\varpi - \psi)i = \frac{1}{\sin\vartheta}\left(i\sqrt{Z} + 2\frac{Lz - L'}{M}\right),$$

so that

$$(7) \qquad \rho^2 = -a\left(z^2 - 4\frac{B}{M}z - 1 - aE\right),$$

(8) $\rho \sin \vartheta \sin(\varpi - \psi) = \sqrt{Z}$,

(9) $\rho \sin \vartheta \cos(\varpi - \psi) = 2\dfrac{Lz - L'}{M}$,

(10) $$\cot(\varpi - \psi) = 2\frac{Lz - L'}{M\sqrt{Z}},$$

(11) $$\varpi = \psi + \cot^{-1} 2\frac{Lz - L'}{M\sqrt{Z}},$$

so that ϖ and ψ depend on the same III elliptic integral, the parameter of which will be denoted by v.

Differentiating

(12) $$\begin{cases} \dfrac{d\varpi}{dt} = qF + 2n\dfrac{-L'z + L}{M(-z^2+1)} + n\sqrt{Z}\dfrac{d}{dz}\cot^{-1} 2\dfrac{Lz - L'}{M\sqrt{Z}} \\ \qquad = qF + 2n\dfrac{\left(z - 2\dfrac{B}{M}\right)\dfrac{Lz - L'}{M}}{z^2 - 4\dfrac{B}{M} - 1 - aE}, \end{cases}$$

(13) $$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{\left(z - 2\dfrac{B}{M}\right)n\sqrt{Z}}{z^2 - 4\dfrac{B}{M}z - 1 - aE},$$

(14) $$\rho\frac{d(\varpi - qFt)}{d\rho} = 2\frac{Lz - L'}{M\sqrt{Z}} = \cot(\varpi - \psi),$$

so that the plane CZ is parallel to the tangent to the curve $(\rho, \varpi - qFt)$; and the velocity along this curve

(15) $$\rho\sqrt{\left[\left(\frac{1}{\rho}\frac{d\rho}{dt}\right)^2 + \left(\frac{d\varpi}{dt} - qF\right)^2\right]} = n\left(z - 2\frac{B}{M}\right)\sin\vartheta.$$

Thus the hodograph of the curve $(\rho, \varpi - qFt)$ is given by

(16) $$\frac{d}{dt}(\rho e^{\varpi i - qFti}) = n\left(z - 2\frac{B}{M}\right)\sin\vartheta e^{\psi i - qFti} = \tfrac{1}{2}an\frac{d\rho^2}{d\vartheta}e^{\psi i - qFti}.$$

The other quantities such as $x_1 + x_2 i$, $y_1 + y_2 i$, $U + Vi$, $P + Qi$ depend on $e^{\phi i}$, and ϕ is given in (31), §4, by two elliptic integrals of the III kind, one with parameter corresponding to $z = \infty$ and denoted by w, the other depending on a parameter denoted by v'.

If v_1 and v_2 are the parameters corresponding to $z = +1$ and -1, equation (32) and (33), §4, show that

(17) $$v = v_1 + v_2, \quad v' = v_1 - v_2,$$

(18) $$v_1 = \tfrac{1}{2}(v + v'), \quad v_2 = \tfrac{1}{2}(v - v').$$

To prove Kirchhoff's expressions for X and Y in (1) and (2), we return to the use of U, V, W, P, Q, R as independent variables, and to the expression of the kinetic energy as a quadratic function of these variables as in §3.

Denoting

(19) $$\cos AX,\ \cos BX,\ \cos CX \text{ by } \alpha_1,\ \alpha_2,\ \alpha_3;$$

(20) $$\cos AY,\ \cos BY,\ \cos CY \text{ by } \beta_1,\ \beta_2,\ \beta_2;$$

(21) $$\cos AZ,\ \cos BZ,\ \cos CZ \text{ by } \gamma_1,\ \gamma_2,\ \gamma_3;$$

then

(22) $$\begin{cases} \dfrac{dX}{dt} = \alpha_1 U + \alpha_2 V + \alpha_3 W \\ \quad = (\beta_2\gamma_3 - \beta_3\gamma_2)U + (\beta_3\gamma_1 - \beta_1\gamma_3)V + (\beta_1\gamma_2 - \beta_2\gamma_1)W \\ \quad = \beta_1(W\gamma_2 - V\gamma_3) + \beta_2(U\gamma_3 - W\gamma_1) + \beta_3(V\gamma_1 - U\gamma_2) \end{cases}$$

and then, since

(23) $$\frac{\partial T}{\partial U} = x_1 = F\gamma_1,\ \frac{\partial T}{\partial V} = x_2 = F\gamma_2,\ \frac{\partial T}{\partial W} = x_3 = F\gamma_3,$$

and (Halphen, F. E. II, p. 148)

(24) $$\frac{d}{dt}\frac{\partial T}{\partial P} = W\frac{\partial T}{\partial V} - V\frac{\partial T}{\partial W} + R\frac{\partial T}{\partial Q} - Q\frac{\partial T}{\partial R}$$

we find

(25) $$\begin{cases} F\dfrac{dX}{dt} = \beta_1\left(W\dfrac{\partial T}{\partial V} - V\dfrac{\partial T}{\partial W}\right) + \beta_2\left(U\dfrac{\partial T}{\partial W} - W\dfrac{\partial T}{\partial U}\right) \\ \qquad\qquad + \beta_3\left(V\dfrac{\partial T}{\partial U} - U\dfrac{\partial T}{\partial V}\right) \\ \quad = \beta_1\dfrac{d}{dt}\dfrac{\partial T}{\partial P} - \beta_1\left(R\dfrac{\partial T}{\partial Q} - Q\dfrac{\partial T}{\partial R}\right) \\ \quad + \beta_2\dfrac{d}{dt}\dfrac{\partial T}{\partial Q} - \beta_2\left(P\dfrac{\partial T}{\partial R} - R\dfrac{\partial T}{\partial P}\right) \\ \quad + \beta_3\dfrac{d}{dt}\dfrac{\partial T}{\partial R} - \beta_3\left(Q\dfrac{\partial T}{\partial P} - P\dfrac{\partial T}{\partial Q}\right) \\ \quad = \beta_1\dfrac{d}{dt}\dfrac{\partial T}{\partial P} + \beta_2\dfrac{d}{dt}\dfrac{\partial T}{\partial Q} + \beta_3\dfrac{d}{dt}\dfrac{\partial T}{\partial R} \\ \quad + (\beta_2 R - \beta_3 Q)\dfrac{\partial T}{\partial P} + (\beta_3 P - \beta_1 R)\dfrac{\partial T}{\partial Q} + (\beta_1 Q - \beta_2 P)\dfrac{\partial T}{\partial R} \\ \quad = \beta_1\dfrac{d}{dt}\dfrac{\partial T}{\partial P} + \beta_2\dfrac{d}{dt}\dfrac{\partial T}{\partial Q} + \beta_3\dfrac{d}{dt}\dfrac{\partial T}{\partial R} \\ \quad + \dfrac{d\beta_1}{dt}\dfrac{\partial T}{\partial P} + \dfrac{d\beta_2}{dt}\dfrac{\partial T}{\partial Q} + \dfrac{d\beta_3}{dt}\dfrac{\partial T}{\partial R} \\ \quad = \dfrac{d}{dt}\left(\beta_1\dfrac{\partial T}{\partial P} + \beta_2\dfrac{\partial T}{\partial Q} + \beta_3\dfrac{\partial T}{\partial R}\right); \end{cases}$$

and integrating

(26)
$$\begin{cases} FX = \beta_1 \dfrac{\partial T}{\partial P} + \beta_2 \dfrac{\partial T}{\partial Q} + \beta_3 \dfrac{\partial T}{\partial R} \\ \quad = \beta_1 y_1 + \beta_2 y_2 + \beta_3 y_3. \end{cases}$$

Similarly

(27)
$$\begin{cases} \dfrac{dY}{dt} = \beta_1 U + \beta_2 V + \beta\ W \\ \quad = (\gamma_2\alpha_3 - \gamma_3\alpha_2)\, U + (\gamma_3\alpha_1 - \gamma_1\alpha_3)\, V + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\, W \\ \quad = -\alpha_1(W\gamma_2 - V\gamma_3) - \alpha_2(U\gamma\alpha_3 - W\gamma_1) - {}_3(V\gamma_1 - U\gamma_2) \end{cases}$$

(28)
$$\begin{cases} F\dfrac{dY}{dt} = -\alpha_1\left(W\dfrac{\partial T}{\partial V} - V\dfrac{\partial T}{\partial W}\right) - \alpha_2\left(U\dfrac{\partial T}{\partial W} - W\dfrac{\partial T}{\partial U}\right) \\ \qquad\qquad - \alpha_3\left(V\dfrac{\partial T}{\partial U} - U\dfrac{\partial T}{\partial V}\right) \\ \quad = -\alpha_1 \dfrac{d}{dt}\dfrac{\partial T}{\partial P} + \alpha_1\left(R\dfrac{\partial T}{\partial Q} - Q\dfrac{\partial T}{\partial R}\right) \\ \qquad - \alpha_2 \dfrac{d}{dt}\dfrac{\partial T}{\partial Q} + \alpha_2\left(P\dfrac{\partial T}{\partial R} - R\dfrac{\partial T}{\partial P}\right) \\ \qquad - \alpha_3 \dfrac{d}{dt}\dfrac{\partial T}{\partial R} + \alpha_3\left(Q\dfrac{\partial T}{\partial P} - P\dfrac{\partial T}{\partial Q}\right) \\ \quad = -\alpha_1 \dfrac{d}{dt}\dfrac{\partial T}{\partial P} - \alpha_2 \dfrac{d}{dt}\dfrac{\partial T}{\partial Q} - \alpha_3 \dfrac{d}{dt}\dfrac{\partial T}{\partial R} \\ \qquad - (\alpha_2 R - \alpha_3 Q)\dfrac{\partial T}{\partial P} - (\alpha_3 P - \alpha_1 R)\dfrac{\partial T}{\partial Q} - (\alpha_1 Q - \alpha_2 P)\dfrac{\partial T}{\partial R} \\ \quad = -\alpha_1 \dfrac{d}{dt}\dfrac{\partial T}{\partial P} - \alpha_2 \dfrac{d}{dt}\dfrac{\partial T}{\partial Q} - \alpha_3 \dfrac{d}{dt}\dfrac{\partial T}{\partial R} \\ \qquad - \dfrac{d\alpha_1}{dt}\dfrac{\partial T}{\partial P} - \dfrac{d\alpha_2}{dt}\dfrac{\partial T}{\partial Q} - \dfrac{d\alpha_3}{dt}\dfrac{\partial T}{\partial R} \\ \quad = -\dfrac{d}{dt}\left(\alpha_1\dfrac{\partial T}{\partial P} + \alpha_2\dfrac{\partial T}{\partial Q} + \alpha_3\dfrac{\partial T}{\partial R}\right); \end{cases}$$

and integrating

(29)
$$\begin{cases} FY = -\alpha_1 \dfrac{\partial T}{\partial P} - \alpha_2 \dfrac{\partial T}{\partial Q} - \alpha_3 \dfrac{\partial T}{\partial R} \\ \quad = -\alpha_1 y_1 - \alpha_2 y_2 - \alpha_3 y_3 \end{cases}$$

(Riemann-Weber, *Partielle Differentialgleichungen*).

As for the coordinate Z (which should be distinguished by an accent from the previous use of Z),

$$(30)\quad \begin{cases} \dfrac{dZ'}{dt} = U\cos AZ + V\cos BZ + W\cos CZ \\ \quad = (px_1 + qy_1)\sin\vartheta\sin\phi + (px_2 + qy_2)\sin\vartheta\cos\phi \\ \qquad\qquad + (p'x_3 + q'y_3)\cos\vartheta) \\ \quad = pF\sin^2\vartheta + p'F\cos^2\vartheta + q\sin\vartheta(y_1\sin\phi + y_2\cos\phi) + q'y_3\cos\vartheta \\ \quad = (p' - p)Fz^2 + pF + \dfrac{q}{r}\sin^2\vartheta\left(\dfrac{d\psi}{dt} - qF\right) + q'y_3 z \\ \quad = (p' - p)Fz^2 + pF + q\left(\dfrac{n'}{F} - y_3 z\right) + q'y_3 z \\ \quad = (p' - p)Fz^2 + (q' - q)y_3 z + pF + \dfrac{qn'}{F} \\ \quad = (p' - p)F\left(z - \dfrac{B}{M}\right)^2 - \tfrac{1}{16}(p' - p)\dfrac{p^2}{F} + pF + \dfrac{qn'}{F} \end{cases}$$

so that Z' depends on an elliptic integral of the second kind, to be given hereafter, in §8.

6. In the special case of $p - p' = 0$,

$$(1)\quad \frac{dz^2}{dt^2} = 2n^2 Z,\ n^2 = (q' - q)Fry_3,$$

$$(2)\quad \begin{cases} Z = (z^2 - 1)(z - D) - 2\left(\dfrac{L'z - L}{M}\right)^2 \\ \quad = (z^2 - 1)(z - E) - 2\left(\dfrac{Lz - L'}{M}\right)^2 \end{cases}$$

and the motion of the axis can be compared directly with that of a symmetrical top moving about its point under gravity, while the curve $(\rho, \varpi - qFt)$ is traced out by the vector of its angular momentum.

Also

$$(3)\quad \frac{dZ'}{dt} = (q' - q)y_3 z + pF + \frac{qn'}{F} = \frac{n^2 z}{Fr} + pF + \frac{qn'}{F}.$$

When $q - q' = 0$, too, the motion is non-elliptic; and when $r - r' = 0$ as well, we have the case "helicoidally isotropic," and the centre describes a uniform helix.

When $m = 0$, $F = 0$, the impulse of the motion reduces to a couple; and

$x_1, x_2, x_3 = 0$,

(4) $$\begin{cases} U = qy_1, & V = qy_2, & W = q'y_3, \\ P = ry_1, & Q = ry_2, & R = r'y_3; \end{cases}$$

and the three equations (3), §1, expressing the constancy in direction of the vector resultant of y_1, y_2, y_3, reduce to

(5) $$\frac{dy_1}{dt} = (r' - r)y_2y_3, \quad \frac{dy_2}{dt} = -(r' - r)y_1y_3, \quad \frac{dy_3}{dt} = 0.$$

Taking OZ in the direction of the resultant couple G,

(6) $y_3 = G \cos \vartheta$, so that ϑ is constant,

(7) $y_1 = G \sin \vartheta \sin \phi = y_3 \tan \vartheta \sin \phi, \quad y_2 = G \sin \vartheta \cos \phi = y_3 \tan \vartheta \cos \phi,$

(8) $\dfrac{d\phi}{dt} = (r' - r)y_3, \quad \cos \vartheta \dfrac{d\psi}{dt} = ry_3,$

(9) $\dfrac{dX}{dt} = -(q - q')y_3 \sin \vartheta \sin \psi, \quad \dfrac{dY}{dt} = (q - q')y_3 \sin \vartheta \cos \psi,$

(10) $X = \dfrac{q - q'}{r} \sin \vartheta \cos \vartheta \cos \psi, \quad Y = \dfrac{q - q'}{r} \sin \vartheta \cos \vartheta \sin \psi,$

(11) $\dfrac{dZ'}{dt} = (q \sin^2 \vartheta + q' \cos^2 \vartheta)\, G, \quad Z' = (q \sin^2 \vartheta - q' \cos^2 \vartheta)\, Gt;$

and the motion of O is helicoidal.

There is an elliptic function solution in the most general case of a body of any shape in the liquid when the resultant impulse of the motion reduces to a couple; the equations of motion reduce to Euler's form for no applied forces, as if the liquid was absent, and a Poinsot geometrical interpretation can be constructed (Lamb, *Proc. London Math. Society*, Vol. VIII; Love, *Proc. Cambridge Phil. Society*, 1889).

7. We now proceed to the Elliptic Function solution, denoting the elliptic argument by u, where

(1) $$u = \int \frac{dz}{\sqrt{Z}},$$

and Z has the form in equation (2), (3), (6), §4.

Employ the Biermann-Weierstrass formula

(2) $$\wp(u_1 \pm u_2) = \frac{F(z_1, z_2) \mp \sqrt{Z_1}\sqrt{Z_2}}{2(z_1 - z_2)^2}$$

where

(3) $F(z_1, z_2) = az_1^2z_2^2 + 2bz_1z_2(z_1 + z_2) + c(z_1^2 + 4z_1z_2 + z_2^2) + 2d(z_1 + z_2) + e.$

Then if v_1 and v_2 correspond to $z_1 = +1$ and $z_2 = -1$,

(4) $$\wp(v_1 \pm v_2) = \frac{1}{8}\left(a - 2c + e \mp 4\frac{L^2 - L'^2}{M^2}\right);$$

so that, putting

(5) $$v_1 + v_2 = v, \quad v_1 - v_2 = v',$$

(6) $$6\wp v = 2a + E - 2\frac{L^2}{M^2},$$

(7) $$6\wp v' = 2a + D - 2\frac{L'^2}{M^2},$$

(8) $$\wp v' - \wp v = \frac{L^2 - L'^2}{M^2} = \tfrac{1}{4}(D - E).$$

Another Biermann-Weierstrass formula gives

(9) $$\left\{\begin{aligned} i\wp'(v_1 \pm v_2) = {} & \frac{L - L'}{M}(-a + 2b - 2d + e) \\ & \mp \frac{L + L'}{M}(-a - 2b + 2d + e), \end{aligned}\right.$$

so that

(10) $$i\wp' v = \frac{EL}{2M} + 2a\frac{BL'}{M^2},$$

(11) $$i\wp' v' = \frac{DL'}{2M} + 2a\frac{BL}{M^2},$$

Again if v_3, v_4 correspond to

(12) $$\rho = 0, \quad z = 2\frac{B}{M} \pm \sqrt{1 + aE + 4\frac{B^2}{M^2}},$$

we shall find

(13) $$6\wp(v_3 + v_4) = 2a + E - 2\frac{L^2}{M^2},$$

so that

(14) $$v_3 + v_4 = v = v_1 + v_2,$$

as could be anticipated from equation (3), §4.

When a root e of the discriminating cubic

(15) $$4s^3 - g_2 s - g_3 = 0$$

is known, and thence the resolution of Z into quadratic factors, say $Z = XY$, then $\wp(u_1 \pm u_2) - e$ is a square, which can be written

(16) $$\wp(u_1 \pm u_2) - e = \tfrac{1}{4}a\left(\frac{\sqrt{X_1}\sqrt{Y_2} \mp \sqrt{X_2}\sqrt{Y_1}}{z_1 - z_2}\right)^2,$$

a formula due to Laguerre (*Bulletin de la Société mathématique*, 1875).

If $u = w$ makes $z = \infty$, then from Hermite's formula, H denoting the Hessian of Z,

(17) $$\wp 2u = \frac{H}{-Z}, \quad H = (ac - b^2)z^4 + \ldots.,$$

(18) $$\wp 2w = \frac{ac - b^2}{-a} = \tfrac{1}{3}a + \tfrac{1}{6}E + \tfrac{2}{3}\frac{L^2}{M^2} + a\frac{B^2}{M^2},$$

(19) $$a^3\wp' 2w = a^2 d - 3abc + 2b^3,$$

(20) $$\wp 2w - \wp v = \frac{aB^2 + L^2}{M^2},$$

(21) $$\wp 2w - \wp v' = \frac{aB^2 + L'^2}{M^2}.$$

Supposing Z is resolved into factors

(22) $$Z = a(z - \alpha)(z - \beta)(z - \gamma)(z - \delta)$$

and that $u = 0$ corresponds to $z = \delta$, the Biermann-Weierstrass formula (2) gives

(23) $$\left\{\begin{aligned} \wp u &= \frac{a\delta^2 z^2 + 2b\delta z(z + \delta) + c(z^2 + 4\delta z + \delta^2) + 2d(z + \delta) + e}{2(z - \delta)^2} \\ &= \frac{a\delta^3 + 3b\delta^2 + 3c\delta + d}{z - \delta} + \tfrac{1}{2}(a\delta^2 + 2b\delta + c) \\ &= \frac{\tfrac{1}{4}Z'(\delta)}{z - \delta} + \tfrac{1}{24}Z''(\delta), \end{aligned}\right.$$

a fundamental formula; and

(24) $$\wp' u = -\frac{\tfrac{1}{4}Z'(\delta)}{(z - \delta)^2}\frac{dz}{du} = -\tfrac{1}{4}Z'(\delta)\frac{\sqrt{Z}}{(z - \delta)^2}.$$

Putting $u = w$, $z = \infty$,

(25) $$\wp w = \tfrac{1}{24}Z''(\delta) = \tfrac{1}{2}(a\delta^2 + 2b\delta + c),$$

(26) $$\wp u - \wp w = \frac{\tfrac{1}{4}Z'(\delta)}{z - \delta} = \frac{a\delta^3 + 3b\delta^2 + 3c\delta + d}{z - \delta},$$

(27) $$\wp' w = -\tfrac{1}{4}Z'(\delta)\sqrt{a},$$

(28) $$\wp' u = \frac{\wp' w}{\sqrt{a}}\frac{\sqrt{Z}}{(z - \delta)^2}$$

and differentiating again

(29) $$\wp'' u = \frac{2\wp' w}{\sqrt{a}}\frac{(a\delta + b)z^2 + (a\delta^2 + 4b\delta + 3c)z + a\delta^3 + 4b\delta^2 + 6c\delta + 3d}{(z - \delta)^2},$$

(30) $$\wp'' w = \frac{2\wp' w}{\sqrt{a}}(a\delta + b).$$

Then

$$\sqrt{a}(z-\delta)=\frac{-\wp' w}{\wp u-\wp w}, \tag{31}$$

and by Laguerre's formula (16)

$$\wp u - e_\alpha = \tfrac{1}{4}a(\delta-\beta)(\delta-\gamma)\frac{(z-\alpha)}{(z-\delta)}, \tag{32}$$

$$\wp w - e_\alpha = \tfrac{1}{4}a(\delta-\beta)(\delta-\gamma), \tag{33}$$

$$\frac{\wp u - e_\alpha}{\wp w - e_\alpha}=\frac{z-\alpha}{z-\delta}, \tag{34}$$

so that

$$\sqrt{a}(z-\alpha)=\frac{-\wp' w}{\wp u-\wp w}\,\frac{\wp u - e_\alpha}{\wp w - e_\alpha}; \tag{35}$$

and by differentiation of (31)

$$\left\{\begin{aligned}\sqrt{a}\frac{dz}{du}&=\sqrt{a}\sqrt{Z}=\frac{\wp' u\wp' w}{(\wp u-\wp w)^2}\\ &=\wp(u-w)-\wp(u+w).\end{aligned}\right. \tag{36}$$

Proceeding with the differentiation

$$\left\{\begin{aligned}az^3+3bz^2+3cz+d&=\tfrac{1}{4}\frac{dZ}{dz}=\tfrac{1}{2}\frac{d\sqrt{Z}}{du}\\ &=\frac{\wp'(u-w)-\wp'(u+w)}{2\sqrt{a}},\end{aligned}\right. \tag{37}$$

$$\left\{\begin{aligned}az^2+2bz+c&=\tfrac{1}{12}\frac{d^2Z}{dz^2}=\frac{1}{6\sqrt{Z}}\frac{d^2\sqrt{Z}}{du^2}\\ &=\tfrac{1}{6}\frac{\wp''(u-w)-\wp''(u+w)}{\wp(u-w)-\wp(u+w)}\\ &=\wp(u-w)+\wp(u+w),\end{aligned}\right. \tag{38}$$

$$\left\{\begin{aligned}(az+b)^2&=a[\wp(u-w)+\wp(u+w)+\wp 2w]\\ &=\tfrac{1}{4}a\left[\frac{\wp'(u-w)+\wp'(u+w)}{\wp(u-w)-\wp(u+w)}\right]^2,\end{aligned}\right. \tag{39}$$

$$\left\{\begin{aligned}az+b&=\tfrac{1}{2}[\wp'(u-w)+\wp'(u+w)]\frac{du}{dz}\\ &=\tfrac{1}{2}\sqrt{a}\,\frac{\wp'(u-w)+\wp'(u+w)}{\wp(u-w)-\wp(u+w)}\\ &=\sqrt{a}[\zeta(u+w)-\zeta(u-w)-\zeta 2w],\end{aligned}\right. \tag{40}$$

and with $z = \delta$, $u = 0$,

(41) $$a\delta + b = \sqrt{a}\,(2\zeta w - \zeta 2w) = \tfrac{1}{2}\sqrt{a}\,\frac{\wp'' w}{\wp' w}.$$

From (25) and (33), by subtraction,

(42) $$e_\alpha = \tfrac{1}{12}a[(\alpha + \delta)(\beta + \gamma) - 2\alpha\delta - 2\beta\gamma]$$

and thence

(43) $$e_\beta - e_\gamma = \tfrac{1}{4}a(\alpha - \delta)(\beta - \gamma).$$

With $z_1 = z$, $u_1 = u$ and $z_2 = \infty$, $u_2 = w$ in the Biermann-Weierstrass formula (2)

(44) $$\wp(u + w) = \tfrac{1}{2}(az^2 + 2bz + c - \sqrt{a}\sqrt{Z}),$$

(45) $$\wp(\omega_\alpha + w) = \tfrac{1}{2}(a\alpha^2 + 2b\alpha + c),$$

supposing $z = \alpha$ makes $u = \omega_\alpha$, a half-period; and then

(46) $$\wp(\omega_\alpha + w) - e_\alpha = \tfrac{1}{4}a(\alpha - \beta)(\alpha - \gamma);$$

and by analogy with (41),

(47) $$a\alpha + b = \tfrac{1}{2}\sqrt{a}\,\frac{\wp''(\omega_\alpha + w)}{\wp'(\omega_\alpha + w)}.$$

8. Treating $\dfrac{FrZ'}{n}$ in (30), §5, by analogy with $\dfrac{Fr}{n}(X + Yi)$ or $\rho e^{\varpi i}$ in (, §5,

(1) $$\begin{cases} \dfrac{d}{du}\left(\dfrac{FrZ'}{n}\right) = \dfrac{d}{dt}\left(\dfrac{FrZ'}{n^2}\right) \\ \qquad = az^2 + 2bz + (mp + n'q)\dfrac{r}{n^2}. \end{cases}$$

Now with

(2) $$Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e,$$

and denoting the Hessian of Z by H,

(3) $$H = \tfrac{1}{12}\frac{d^2Z}{dz^2}Z - \left(\tfrac{1}{4}\frac{dZ}{dz}\right)^2,$$

while

(4) $$\frac{d^2\sqrt{Z}}{dz^2} = \tfrac{1}{2}\frac{d^2Z}{dz^2}\frac{1}{\sqrt{Z}} - \tfrac{1}{4}\left(\frac{dZ}{dz}\right)^2\frac{1}{Z\sqrt{Z}},$$

so that

(5) $$-\frac{2H}{Z\sqrt{Z}} + \tfrac{1}{2}\frac{d^2\sqrt{Z}}{dz^2} = \tfrac{1}{12}\frac{d^2Z}{dz^2}\frac{1}{\sqrt{Z}} = \frac{az^2 + 2bz + c}{\sqrt{Z}},$$

and

(6) $$\frac{FrZ'}{n} = -2\int \frac{H}{Z}\frac{dz}{\sqrt{Z}} + \tfrac{1}{2}\frac{d\sqrt{Z}}{dz} - cu + (mp + n'q)\frac{ru}{n^2},$$

in which

(7) $$\frac{H}{Z} = -\wp 2u, \text{ so that } -2\int \frac{H}{Z}\frac{dz}{\sqrt{Z}} = -\zeta 2u,$$

while

(8) $$\begin{cases} \tfrac{1}{2}\dfrac{d\sqrt{Z}}{dz} = \tfrac{1}{2}\dfrac{d\sqrt{Z}}{du}\Big/\dfrac{dZ}{du} = \tfrac{1}{2}\dfrac{\wp'(u-w) - \wp'(u+w)}{\wp(u-w) - \wp(u+w)} \\ \qquad = \zeta 2u - \zeta(u-w) - \zeta(u+w), \end{cases}$$

(9) $$\frac{FrZ'}{n} = \left(\frac{mpr + n'qr}{n^2} - c\right)u - \zeta(u-w) - \zeta(u+w),$$

the form to employ when $a = +1$, and w is a fraction of the real period. Otherwise with $a = -1$ it is preferable to return to (6) and (7); and then, in the reduction to the Jacobian function of the II stage, where the roots e_1, e_2, e_3 of the discriminating cubic (15), §7, are known, we find

(10) $$\begin{cases} \displaystyle\int (az^2 + 2bz + c)\frac{dz}{\sqrt{Z}} = -2(e_1 - e_3)\int \frac{H + e_2 Z}{H + e_3 Z}\frac{dz}{\sqrt{Z}} \\ \qquad\qquad + \tfrac{1}{4}\sqrt{Z}\dfrac{d}{dz}\log(H + e_3 Z) + 2e_1\displaystyle\int \frac{dz}{\sqrt{Z}}, \end{cases}$$

as is verified by differentiation.

With the roots in the order $e_1 > e_2 > e_3$,

(11) $$\frac{H + e_2 Z}{H + e_3 Z} = \frac{\wp 2u - e_2}{\wp 2u - e_3} = \mathrm{dn}^2 2Mu, \quad M^2 = e_1 - e_3,$$

(12) $$2(e_1 - e_3)\int \frac{H + e_2 Z}{H + e_3 Z}\frac{dz}{\sqrt{Z}} = 2M^2 \int \mathrm{dn}^2 2Mu\,du = 2M^2 \frac{E}{K}u + M\,\mathrm{zn}\,2Mu,$$

so that $\dfrac{FrZ'}{n}$ is expressed by secular terms proportional to u or the time, by an algebraical function $\tfrac{1}{4}\sqrt{Z}\dfrac{d}{dz}\log(H + e_3 Z)$, and by the zeta-function $M\,\mathrm{zn}\,2Mu$, functions which do not become infinite.

9. With $u = v_1$, v_2 for $z = +1, -1$,

(1) $$\sqrt{a}(1 - z_0) = \frac{-\wp' w}{\wp v_1 - \wp w},$$

(2) $$\sqrt{a}(1 - z) = \frac{-\wp' w(\wp u - \wp v_1)}{(\wp v_1 - \wp w)(\wp u - \wp w)},$$

and similarly

(3) $$\sqrt{a}(1+z)=\frac{\wp' w(\wp u-\wp v_2)}{(\wp v_2-\wp w)(\wp u-\wp w)}.$$

Also

(4) $$2i\sqrt{a}\,\frac{L-L'}{M}=\frac{-\wp' v_1\wp' w}{(\wp v_1-\wp w)^2}=\wp(v_1+w)-\wp(v_1-w),$$

(5) $$2i\sqrt{a}\,\frac{L+L'}{M}=\frac{\wp' v_2\wp' w}{(\wp v_2-\wp w)^2}=\wp(v_2-w)-\wp(v_2+w),$$

with

(6) $$\psi-qFt=\psi_1+\psi_2,$$

(7) $$\psi_1=\frac{L-L'}{M}\int\frac{du}{1-z},$$

(8) $$\psi_2=\frac{L+L'}{M}\int\frac{du}{1+z},$$

(9) $$\left\{\begin{aligned}\frac{d\psi_1 i}{du}&=\frac{\dfrac{1}{2\sqrt{a}}\,\dfrac{-\wp' v_1\wp' w}{(\wp v_1-\wp w)^2}}{\dfrac{1}{\sqrt{a}}\,\dfrac{-\wp' w(\wp u-\wp v_1)}{(\wp v_1-\wp w)(\wp u-\wp w)}}\\&=\tfrac{1}{2}\,\frac{\wp' v_1(\wp u-\wp w)}{(\wp v_1-\wp w)(\wp u-\wp v_1)}\\&=\frac{\tfrac{1}{2}\wp' v_1}{\wp v_1-\wp w}+\frac{\tfrac{1}{2}\wp' v_1}{\wp u-\wp v_1}\\&=\tfrac{1}{2}\zeta(v_1-w)+\tfrac{1}{2}\zeta(v_1+w)-\zeta v_1\\&\quad+\tfrac{1}{2}\zeta(u-v_1)-\tfrac{1}{2}\zeta(u+v_1)+\zeta v_1.\end{aligned}\right.$$

Similarly

(10) $$\frac{d\psi_2 i}{du}=\tfrac{1}{2}\zeta(v_2-w)+\tfrac{1}{2}\zeta(v_2+w)-\zeta v_2+\tfrac{1}{2}\zeta(u-v_2)-\tfrac{1}{2}\zeta(u+v_2)+\zeta v_2.$$

Integrating

(11) $$\psi_1 i=\tfrac{1}{2}[\zeta(v_1-w)+\zeta(v_1+w)]\,nt+\tfrac{1}{2}\log\frac{\sigma(u-v_1)}{\sigma(u+v_1)},$$

(12) $$\psi_2 i=\tfrac{1}{2}[\zeta(v_2-w)+\zeta(v_2+w)]\,nt+\tfrac{1}{2}\log\frac{\sigma(u-v_2)}{\sigma(u+v_2)}.$$

With $v_1+v_2=v$, and the formula

(13) $$\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)}=\frac{\sigma(2u-v)\sigma^2(u+v_1)\sigma^2(u+v_2)}{\sigma(2u+v)\sigma^2(u-v_1)\sigma^2(u-v_2)},$$

(14) $$\psi_1 i + \psi_2 i = \tfrac{1}{2} Qnt + \tfrac{1}{2}\log\frac{\sigma(2u-v)}{\sigma(2u+v)} - \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)},$$

(15) $$Q = \zeta(v_1-w)+\zeta(v_1+w)+\zeta(v_2-w)+\zeta(v_2+w).$$

Introducing a standard form of the III elliptic integral from the memoir on the subject in the *Phil. Trans.*, 1904,

(16) $$\begin{cases} I(2u,v) = \displaystyle\int \frac{\frac{P}{M}(s-\sigma)-\frac{1}{2}\surd-\Sigma}{s-\sigma}\frac{ds}{\surd S} \\ \qquad = \tfrac{1}{2}i\left\{\log\dfrac{\sigma(2u+v)}{\sigma(2u-v)} - \left(\dfrac{Pi}{M}+\zeta v\right)2u\right\}, \end{cases}$$

(17) $$2u = \int\frac{ds}{\surd S},$$

then

(18) $$\begin{cases} (\psi - qFt)i = (\psi_1+\psi_2)i \\ \qquad = \tfrac{1}{2}\left(Q-2\zeta v-2\dfrac{Pi}{M}\right)nt + \tfrac{1}{2}iI(2u,v) - \tfrac{1}{4}\log\dfrac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)}. \end{cases}$$

Another application of the Biermann-Weierstrass formula (2), §7, gives

(19) $$\wp(u\pm v_1) = \frac{(a+2b+c)z^2+2(b+2c+d)z+c+2d+e\pm 2i\dfrac{L-L'}{M}\surd Z}{2(z-1)^2}$$

(20) $$\wp(u\pm v_2) = \frac{(a-2b+c)z^2+2(b-2c+d)z+c-2d+e\pm 2i\dfrac{L+L'}{M}\surd Z}{2(z+1)^2}$$

(21) $$\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \frac{A'+iB'\surd Z}{A'-iB'\surd Z},$$

(22) $$\tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \tfrac{1}{2}i\tan^{-1}\frac{B'\surd Z}{A'},$$

(23) $$A' = bz^4+(a+3c)z^3+3(b+d)z^2+(3c+e)z+d,$$

(24) $$B' = \frac{Lz^2-2L'z+L}{M}.$$

A further application of formula (44), §7, gives

(25) $$\wp(v_1\pm w) = \frac{a+2b+c}{2}\pm 2i\surd a\,\frac{L-L'}{M},$$

(26) $$\wp(v_2\pm w) = \frac{a-2b+c}{2}\mp 2i\surd a\,\frac{L+L'}{M},$$

(27) $$\wp(v_1+w)+\wp(v_2-w)=a+c+2i\sqrt{a}\,\frac{L}{M},$$

(28) $$\left\{\begin{aligned}\wp(v_1-w)+\wp(v_2+w)&=a+c-2i\sqrt{a}\,\frac{L}{M}\\&=-\tfrac{2}{3}\frac{L^2}{M^2}+\tfrac{2}{3}a-\tfrac{1}{6}E-2i\sqrt{a}\,\frac{L}{M},\end{aligned}\right.$$

while

(29) $$\wp(v_1+v_2)=-\frac{L^2}{M^2}+\tfrac{1}{3}a+\tfrac{1}{6}E,$$

and then

(30) $$\begin{aligned}\wp(v_1+w)+\wp(v_2-w)+\wp(v_1+v_2)&=-\frac{L^2}{M^2}+2i\sqrt{a}\,\frac{L}{M}+a\\&=\left(i\frac{L}{M}+\sqrt{a}\right)^2,\end{aligned}$$

(31) $$\wp(v_1-w)+\wp(v_2+w)+\wp(v_1+v_2)=\left(i\frac{L}{M}-\sqrt{a}\right)^2,$$

(32) $$\zeta(v_1+w)+\zeta(v_2-w)-\zeta(v_1+v_2)=i\frac{L}{M}+\sqrt{a},$$

(33) $$\zeta(v_1-w)+\zeta(v_2+w)-\zeta(v_1+v_1)=i\frac{L}{M}-\sqrt{a},$$

(34) $$\left\{\begin{aligned}Q-2\zeta v&=\zeta(v_1+w)+\zeta(v_2-w)-\zeta(v_1+v_2)\\&\quad+\zeta(v_1-w)+\zeta(v_2+w)-\zeta(v_1+v_2)\\&=2i\frac{L}{M},\end{aligned}\right.$$

so that finally,

(35) $$\psi-qFt=\frac{L-P}{M}nt+\tfrac{1}{2}I(2u,v)+\tfrac{1}{2}\tan^{-1}\frac{B'\sqrt{Z}}{A'}.$$

Putting

(36) $$\psi-qFt-\frac{L-P}{M}nt=\psi-pt=\psi',$$

(37) $$\frac{p}{n}=\frac{qF}{n}+\frac{L-P}{M},$$

(38) $$\left\{\begin{aligned}\psi'&=2\int\frac{L-L'z}{M(1-z^2)}\frac{dz}{\sqrt{Z}}-\frac{L-P}{M}\int\frac{dz}{\sqrt{Z}}\\&=\int\frac{(L-P)z^2-2L'z+L+P}{M(1-z^2)}\frac{dz}{\sqrt{Z}}\\&=\tfrac{1}{2}I(2u,v)+\tfrac{1}{2}\tan^{-1}\frac{B'\sqrt{Z}}{A'},\end{aligned}\right.$$

and the right-hand side can be made the logarithm of an algebraical function of z when v is chosen as an aliquot μth part of a period, as shown already in the *American Journal of Mathematics*, XX, 1897; and for subsequent details consult the *Phil. Trans.*, 1904, on the Third Elliptic Integral, etc.; to which references in the sequel are indicated by the page and article or section.

For comparison with Halphen's results the following table gives his notation in the left-hand column, and its equivalent on the right as employed in this memoir.

u	$u - w$
$u + v$	$u + w$
v	$2w$
a	$v_1 - w$
b	$-v_2 - w$
$a - b$	v
$a + b + v$	v'
$a + v$	$v_1 + w$
$b + v$	$-v_2 + w$
$u - a$	$u - v_1$
$u + a + v$	$u + v_1$
a_1	$-v_3 - w$
b_1	$v_4 - w$
$-a_1 + b_1$	v
c	$\frac{1}{2}(v_1 - v_3) - w$
c	$-\frac{1}{2}(v_2 - v_4) - w$
a'	$-\frac{1}{2}(v_1 + v_3)$
b'	$\frac{1}{2}(v_2 + v_4)$
w	$\frac{1}{2}(v_1 - v_3) + w$
w	$-\frac{1}{2}(v_2 - v_4) + w$

10. Denoting Klein's functions by α_1, β_1, γ_1, δ_1, in Kirchhoff's special case where q, $q' = 0$, so that

(1) $$\tfrac{1}{2}(\phi + \psi) = \psi_2, \quad \tfrac{1}{2}(\phi - \psi) = -\psi_1,$$

then

(2) $$\alpha_1 = \sqrt{\frac{1 + z}{2}}\, e^{\psi_2 i},$$

$$(3)\quad \begin{cases} \log \alpha_1 = \psi_2 i + \log \sqrt{\dfrac{1+z}{2}} \\ \quad = \frac{1}{2}[\zeta(v_2 - w) + \zeta(v_2 + w)]u \\ \qquad + \frac{1}{2}\log \dfrac{\sigma(u - v_2)}{\sigma(u + v_2)} \dfrac{\frac{1}{2}\wp' w(\wp u - \wp v_2)}{\sqrt{a(\wp v_2 - \wp w)(\wp u - \wp w)}} \\ \quad = \frac{1}{2}[\zeta(v_2 - w) + \zeta(u_2 + w) - 2\zeta v_2]u \\ \qquad + \log \dfrac{\sigma(u - v_2)}{\sigma u \sigma v_2} e^{u\zeta v_2} \sqrt{\dfrac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_2 - \wp w)(\wp u - \wp w)}}}, \end{cases}$$

$$(4)\qquad \alpha_1 = e^{l_2 nti} \frac{\sigma(u - v_2)}{\sigma u \sigma v_2} e^{u\zeta v_2} \sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_2 - \wp w)(\wp u - \wp w)}}},$$

where

$$(5)\qquad l_2 = -\tfrac{1}{2}[\zeta(v_2 - w) + \zeta(v_2 + w) - 2\zeta v_2]i = \frac{-\frac{1}{2}i\wp' v_2}{\wp v_2 - \wp w}.$$

Similarly

$$(6)\qquad \beta_1 = \sqrt{\frac{-1+z}{2}} e^{\psi_1 i} = e^{l_1 nti} \frac{\sigma(u - v_1)}{\sigma u \sigma v_1} e^{u\zeta v_1} \sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_1 - \wp w)(\wp u - \wp w)}}},$$

where

$$(7)\qquad l_1 = \frac{-\frac{1}{2}i\wp' v_1}{\wp v_1 - \wp w},$$

and

$$(8)\qquad \gamma_1 = e^{-l_1 nti} \frac{\sigma(u + v_1)}{\sigma u \sigma v_1} e^{-u\zeta v_1} \sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_1 - \wp w)(\wp u - \wp w)}}},$$

$$(9)\qquad \delta_1 = e^{-l_2 nti} \frac{\sigma(u + v_2)}{\sigma u \sigma v_2} e^{-u\zeta v_2} \sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_2 - \wp w)(\wp u - \wp w)}}}.$$

When μv_1 or μv_2 is congruent to a period, and μ is an odd integer $= 2n + 1$ then α_1, β_1, γ_1, or δ_1, is the $(2n + 1)$th root of algebraical functions of the form

$$(10)\qquad A + iB\sqrt{Z},$$

such that

$$(11)\qquad A^2 + B^2 Z = \left(\frac{\pm 1 + z}{2}\right)^{2n+1},$$

and A, B can be determined by the method of *réduites* (Halphen, F. E. II, Chap. 14) or else as explained in *Phil. Trans.*, 1904.

But when μ is even $= 2n$, and nv_1 or nv_2 is congruent to a half-period then α_1, β_1, γ_1, or δ_1, is the nth root of an algebraical function of the form

$$(12)\qquad A_1\sqrt{Z_1} + iA_2\sqrt{Z_2},$$

such that

$$(13) \qquad A_1^2 Z_1 + A_2^2 Z_2 = \left(\frac{\pm 1 + z}{2}\right)^n,$$

Z_1 and Z_2 denoting the quadratic factors of Z.

In Clebsch's case where q and q' are not zero, then Klein's functions are given in general by

$$(14) \qquad \alpha = e^{iA}\alpha_1, \quad \beta = e^{-iA'}\beta_1, \quad \gamma = e^{iA'}\gamma_1, \quad \delta = e^{-iA}\delta_1,$$

where from equations (32), (33), §4,

$$(15) \qquad A, A' = \tfrac{1}{2}\frac{F}{n}\int [(q'-q)z + (r'-r)\frac{y_3}{F} \pm q]\,du,$$

and introducing Halphen's notation (F. E. II, p. 157)

$$(16) \qquad \frac{(q'-q)F}{\sqrt{an^2}} = \frac{q'-q}{\sqrt{n(p'-p)}} = -\frac{B}{L}\sqrt{a} = -\beta i,$$

$$(17) \qquad A = -\tfrac{1}{2}\beta i \int \sqrt{a}\left(z - \frac{B}{M}\right)du - \tfrac{1}{2}\beta\sqrt{(-a)}\frac{Bu}{M} + (r'-r)\frac{y_3 u}{2n} + \frac{Fqu}{2n},$$

in which

$$(18) \qquad \left\{ \begin{aligned} \int \sqrt{a}\left(z - \frac{B}{M}\right)du &= \int [\zeta(u+w) + \zeta(u-w) - \zeta 2w]\,du \\ &= \log \frac{\sigma(u+w)}{\sigma(u-w)} e^{-u\zeta 2w}, \end{aligned} \right.$$

so that

$$(19) \qquad e^{iA} = \left[\frac{\sigma(u+w)}{\sigma(u-w)} e^{-u\zeta 2w}\right]^{\frac{1}{2}\beta} e^{Mui},$$

is the form of the result for A and for A' also; and if $q = q'$, $B = 0$, A and A' reduce to a multiple of the time.

11. As the simplest illustration, suppose v is a half-period, so that

(1) $$\wp' v = 0, \quad \frac{EL}{M} + 4a\frac{BL'}{M^2} = 0,$$

from (10), §7, and then from (3), §4,

(2) $$Z = a(z^2-1)^2 - 4a\frac{B}{L}(z^2-1)\frac{Lz-L'}{M} - 4\left(\frac{Lz-L'}{M}\right)^2;$$

and

(3) $$k^2(z^2-1)^2 - Z = (k^2-a)(z^2-1)^2 + 4a\frac{B}{L}(z^2-1)\frac{Lz-L'}{M} + 4\left(\frac{Lz-L'}{M}\right)^2$$

is made a square if

(4) $$k^2 - a = \frac{B^2}{L^2}, \quad k^2 = \frac{B^2+aL^2}{L^2}.$$

Putting

(5) $$\left\{\begin{aligned} \psi' &= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{Z}}{k(1-z^2)} \\ &= \tfrac{1}{2}\cos^{-1}\frac{a\dfrac{B}{L}(1-z^2) + 2\dfrac{Lz-L'}{M}}{k(1-z^2)} \\ &= \frac{\cos^{-1}}{\sin^{-1}}\sqrt{\frac{\{\sqrt{(B^2+aL^2)} \pm aB\}(1-z^2) \pm 2L\dfrac{Lz-L'}{M}}{2\sqrt{(B^2+aL^2)}(1-z^2)}}, \end{aligned}\right.$$

(6) $$\left\{\begin{aligned} \frac{d\psi'}{dz} &= \frac{2\dfrac{L-L'z}{M} - \dfrac{L}{M}(1-z^2)}{(1-z^2)\sqrt{Z}} \\ &= \frac{d\psi}{dz} - qF\frac{dt}{dz} - \frac{Ln}{M}\frac{dt}{dz}, \end{aligned}\right.$$

(7) $$\psi - qFt - \frac{L}{M}nt = \psi'.$$

In Kirchhoff's case of $B = 0$, $a = -1$ is to be rejected, and

(8) $$Z_1, Z_2 = z^2 \mp 2\frac{L'z}{M} - 1 \pm 2\frac{L}{M}.$$

Again

(9) $$\rho^2 = -a\left(z^2 - 4\frac{B}{M}z - 1 + 4\frac{B}{M}\frac{L'}{L}\right),$$

(10) $$z^2 - 1 = -4\frac{B}{L}\frac{L'-Lz}{M} - a\rho^2,$$

(11) $$Z = \rho^2\left(a\rho^2 + 4\frac{B}{L}\frac{L'-Lz}{M}\right) - 4\left(\frac{L'-Lz}{M}\right)^2,$$

(12) $$\lambda^2\rho^4 - Z = (\lambda^2 - a)\rho^4 - 4\frac{B}{L}\frac{L'-Lz}{M}\rho^2 + 4\left(\frac{L'-Lz}{M}\right)^2,$$

is a square if

(13) $$\lambda^2 - a = \frac{B^2}{L^2}, \quad \lambda = k;$$

and then putting

(14) $$\varpi' = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{Z}}{k\rho^2} = \tfrac{1}{2}\cos^{-1}\frac{a\frac{B}{L}\rho^2 - 2\frac{L'-Lz}{M}}{k\rho^2},$$

(15) $$\frac{d\varpi'}{dz} = -\frac{2a\left(z - 2\frac{B}{M}\right)\frac{Lz-L'}{M}}{\rho^2\sqrt{Z}} - \frac{L}{M}\frac{1}{\sqrt{Z}} = \frac{d\varpi}{dz} - qF\frac{dt}{dz} - \frac{Ln}{M}\frac{dt}{dz},$$

(16) $$\varpi - qFt - \frac{L}{M}nt = \varpi'.$$

In this case the vector $\sin\vartheta e^{\psi' i}$ and $\rho e^{\varpi' i}$ each describes the arc of a conic on a plane perpendicular to OZ, and the motion is algebraical except for the secular term $qFt + \frac{L}{M}nt$, and this can be cancelled by putting

(17) $$qFM + Ln = 0.$$

Now suppose in addition that v' is also a half-period so that

(18) $$v_1, v_2 = \omega_1 + \tfrac{1}{2}\omega_3, \tfrac{1}{2}\omega_3,$$

(19) $$\wp' v' = 0, \quad \frac{DL'}{M} + 4a\frac{BL}{M^2} = 0, \text{ from (11), §7},$$

(20) $$\frac{D}{E} = \frac{L^2}{L'^2}, \quad D - E = 4\frac{L^2 - L'^2}{M^2},$$

(21) $$D = 4\frac{L^2}{M^2}, \quad E = 4\frac{L'^2}{M^2}, \quad aB = -\frac{LL'}{M}.$$

(22) $$\begin{cases} Z = a(z^2-1)\left(z^2 + 4a\frac{LL'}{M^2}z - 1 - 4a\frac{L'^2}{M^2}\right) - 4\left(\frac{Lz-L'}{M}\right)^2 \\ \quad = a\left(z^2 + 2a\frac{LL'}{M} + 1\right)^2 - 4a\left(\frac{L^2}{M^2} + a\right)\left(\frac{L'^2}{M^2} + a\right)z^2. \end{cases}$$

Writing λ, λ' for $\frac{L, L'}{M}$,

(23) $$aZ = (z^2 + 2a\lambda\lambda' z + 1)^2 - 4(\lambda^2 + a)(\lambda'^2 + a)z^2.$$

With $a = 1$, put $\lambda, \lambda' = \operatorname{sh} \alpha, \operatorname{sh} \alpha'$; then

(24) $$Z = Z_1 Z_2,$$

(25) $$Z_1 = z^2 + 2z \operatorname{ch}(\alpha + \alpha') + 1,$$ (26) $$Z_2 = z^2 - 2z \operatorname{ch}(\alpha - \alpha') + 1;$$

and with $\alpha > \alpha'$, the roots arranged in ascending order are

(27) $$-e^{\alpha+\alpha'}, \quad -e^{-\alpha-\alpha'}, \quad e^{-\alpha+\alpha'}, \quad e^{\alpha-\alpha'};$$

and denoting then by z_0, z_3, z_2, z_1,

(28) $$Z_1 = (z - z_0)(z - z_3),$$ (29) $$Z_2 = (z_2 - z)(z_1 - z),$$
and

(30) $$z_0 < -1 < z_3 < z < z_2 < 1 < z_1,$$

With $a = -1$, putting $\lambda, \lambda' = \operatorname{ch} \alpha, \operatorname{ch} \alpha'$,

(31) $$Z_1 = z_2 - 2z \operatorname{ch}(\alpha - \alpha') + 1, \quad Z_2 = -z^2 + 2z \operatorname{ch}(\alpha + \alpha') - 1,$$

and arranged in ascending order

(32) $$e^{-\alpha-\alpha'}, \quad e^{-\alpha+\alpha'}, \quad e^{\alpha-\alpha'}, \quad e^{\alpha+\alpha'},$$
denoted by z_3, z_2, z_1, z_0,

(33) $$Z_1 = (z_1 - z)(z_2 - z),$$ (34) $$Z_2 = (z_0 - z)(z - z_3),$$

(35) $$-1 < z_3 < z < z_2 < 1 < z_1 < z_0.$$

With $a = 1$, we can make

(36) $$A_1^2 Z_1 + A_2^2 Z_2 = \left(\frac{z \pm 1}{2}\right)^2,$$

by taking

(37) $$A_1 = \frac{\operatorname{sh} \frac{1}{2}(\alpha + \alpha') \text{ or } \operatorname{ch} \frac{1}{2}(\alpha + \alpha')}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}},$$ (38) $$A_2 = \frac{\operatorname{ch} \frac{1}{2}(\alpha - \alpha') \text{ or } \operatorname{sh} \frac{1}{2}(\alpha - \alpha')}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}},$$

and then put

(39) $$\alpha_1 = \left[\frac{\operatorname{sh} \frac{1}{2}(\alpha + \alpha')\sqrt{Z_1} + i \operatorname{ch} \frac{1}{2}(\alpha - \alpha')\sqrt{Z_2}}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}}\right]^{\frac{1}{2}},$$

(40) $$\beta_1 = \left[\frac{\operatorname{ch} \frac{1}{2}(\alpha + \alpha')\sqrt{Z_1} + i \operatorname{sh} \frac{1}{2}(\alpha - \alpha')\sqrt{Z_2}}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}}\right]^{\frac{1}{2}},$$

Then Klein's $\alpha, \beta, \gamma, \delta$, are of the form

(41) $$\alpha = e^{l_2 nti} G^{\frac{1}{2}\beta i} \alpha_1,$$ (42) $$\beta = e^{l_1' nti} G^{-\frac{1}{2}\beta i} \beta_1,$$ (43) $$\gamma = e^{-l_1' nti} G^{\frac{1}{2}\beta i} \gamma_1,$$

(44) $$\delta = e^{-l_2 nti} G^{-\frac{1}{2}\beta i} \delta_1;$$

and

(45) $\frac{1}{2}i \sin \vartheta e^{\psi i} = \alpha\beta = e^{(l_1+l_2)nti}\alpha_1\beta_1,$

(46) $\wp v' = e_1 = \frac{2}{3}\,\mathrm{sh}^2\,\alpha - \frac{1}{3}\,\mathrm{sh}^2\,\alpha' + \frac{1}{3},$

(47) $\wp v = e_2 = -\frac{1}{3}\,\mathrm{sh}^2\,\alpha + \frac{2}{3}\,\mathrm{sh}^2\,\alpha' + \frac{1}{3},$

(48) $e_3 = -\frac{1}{3}\,\mathrm{sh}^2\,\alpha - \frac{1}{3}\,\mathrm{sh}^2\,\alpha' - \frac{2}{3},$

(49) $e_1 - e_2 = \mathrm{sh}^2\,\alpha - \mathrm{sh}^2\,\alpha',$ (50) $e_1 - e_3 = \mathrm{ch}^2\,\alpha,$ (51) $e_2 - e_3 = \mathrm{ch}^2\,\alpha',$

(52) $\wp 2w = -c + \dfrac{b^2}{a} = \frac{2}{3}\,\mathrm{sh}^2\,\alpha + \frac{2}{3}\,\mathrm{sh}^2\,\alpha' + \frac{1}{3} + \mathrm{sh}^2\,\alpha\,\mathrm{sh}^2\,\alpha',$

(53) $\wp 2w - e_1 = \mathrm{ch}^2\alpha\,\mathrm{sh}^2\alpha',$ (54) $\wp 2w - e_2 = \mathrm{sh}^2\alpha\,\mathrm{ch}^2\alpha',$ (55) $\wp 2w - e_3 = \mathrm{ch}^2\alpha\,\mathrm{ch}^2\alpha',$

(56) $\wp w - e_1 = \mathrm{sh}(\alpha + \alpha')\,\mathrm{ch}\,\alpha(\mathrm{ch}\,\alpha' + \mathrm{sh}\,\alpha'),$

(57) $\wp w - e_2 = \mathrm{sh}(\alpha + \alpha')\,\mathrm{ch}\,\alpha'(\mathrm{ch}\alpha + \mathrm{sh}\,\alpha),$

(58) $\wp w - e_3 = \mathrm{ch}\,\alpha\,\mathrm{ch}\,\alpha'[\mathrm{ch}(\alpha + \alpha') + \mathrm{sh}(\alpha + \alpha')].$

Try

(59) $L = L', \quad a = a', \quad D = E = -4a\dfrac{B}{M}, \quad BM = -aL^2,$

(60) $Z = a(z-1)^2\left[(z+1)^2 + 4a\dfrac{L^2}{M^2}z\right].$

representing a state of steady motion.

Take $a = +1$,

(61) $Z_1 = (z-1)^2, \quad Z_2 = z^2 + 2z\,\mathrm{ch}\,2a + 1,$

(62) $\begin{matrix}\alpha_1\\ \delta_1\end{matrix} = \left[\dfrac{\mathrm{sh}\,a(z-1) \pm i\sqrt{Z_2}}{2\,\mathrm{ch}\,a}\right]^{\frac{1}{2}}, \quad \alpha_1\delta_1 = \dfrac{z+1}{2},$

(63) $\beta_1 = \gamma_1 = \left(\dfrac{z-1}{2}\right)^{\frac{1}{2}}, \quad \beta_1\gamma_1 = \dfrac{z-1}{2}, \quad \psi_1 = 0.$

Take $a = -1$,

(64) $Z_1 = (z-1)^2, \quad Z_2 = -z^2 + 2z\,\mathrm{ch}\,2a - 1,$

(65) $\begin{matrix}\alpha_1\\ \delta_1\end{matrix} = \left(\dfrac{\mathrm{ch}\,a(z-1) \pm i\sqrt{Z_2}}{2\,\mathrm{sh}\,a}\right)^{\frac{1}{2}}, \quad \alpha_1\delta_1 = \dfrac{z+1}{2},$

(66) $\beta_1 = \gamma_1 = \left(\dfrac{z-1}{2}\right)^{\frac{1}{2}}, \quad \beta_1\gamma_1 = \dfrac{z-1}{2}, \quad \psi_1 = 0.$

[*To be continued.*]